Due Sat

4.3 - Homogeneous Linear Equations with Constant Coefficients

Ex: $y'' - 5y' + 6y = 0$

Assume a solution of the form $y = e^{mx}$. Then $y' = me^{mx}$ and $y'' = m^2 e^{mx}$.

We get $m^2 e^{mx} - 5m e^{mx} + 6e^{mx} = 0$

$e^{mx}(m^2 - 5m + 6) = 0 \quad (e^{mx} \neq 0)$

$m^2 - 5m + 6 = 0$ auxiliary equation

$(m-3)(m-2) = 0 \Rightarrow m = 3, 2$

$y_1 = e^{3x}, \; y_2 = e^{2x}$

$$\boxed{y = c_1 e^{3x} + c_2 e^{2x}}$$

Def: The **auxiliary equation** of the second-order linear differential equation $ay'' + by' + cy = 0$ is $am^2 + bm + c = 0$.

The auxiliary equation has solutions

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

this leads to 3 cases:

Case 1: $m_1 \neq m_2$, both are real.

Then $\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$

Case 2: $m_1 = m_2 = m$ (real)

$(b^2 - 4ac = 0) \Rightarrow m = -\frac{b}{2a}$

$y_1 = e^{-b/2a\, x}$

$ay'' + by' + cy = 0$

$P(x) = \frac{b}{a}$

Using reduction of order:

$$y_2 = e^{-b/2a\, x} \int \frac{e^{-\int \frac{b}{a} dx}}{e^{-b/a\, x}} dx$$

$$y_2 = e^{-b/2a\, x} \int dx = e^{-b/2a\, x} \cdot x$$

Then $y_2 = x e^{mx}$

and $\boxed{y = c_1 e^{mx} + c_2 x e^{mx}}$

Case 3: $b^2-4ac<0 \Rightarrow m_1=\alpha+\beta i,\ m_2=\alpha-\beta i$

$e^{\alpha x}\boxed{e^{\beta i x}}$ $\quad$ $e^{\alpha x}\boxed{e^{-\beta i x}}$

$$y = c_1 e^{(\alpha+\beta i)x} + c_2 e^{(\alpha-\beta i)x}$$

$$e^{\alpha x}\left(\boxed{c_1 e^{\beta i x} + c_2 e^{-\beta i x}}\right)$$

Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$\theta = \pi$

$e^{i\pi} = -1$

$e^{i\pi}+1=0$

So $e^{i\beta x} = \cos\beta x + i\sin\beta x$

$e^{-i\beta x} = \cos\beta x - i\sin\beta x$

$$e^{i\beta x}+e^{-i\beta x} = 2\cos\beta x$$

$$\Rightarrow \cos\beta x = \frac{1}{2}\left(e^{\beta x i}+e^{-\beta x i}\right)$$

$c_1=c_2=\frac{1}{2}$

↳ cosine

Likewise, $\sin\beta x = \frac{1}{2i}\left(e^{\beta x i}-e^{-\beta x i}\right)$

$c_1=\frac{1}{2i},$

$c_2=-\frac{1}{2i}$

↳ sine

$$y_1 = e^{\alpha x}\cos\beta x, \quad y_2 = e^{\alpha x}\sin\beta x$$

Solution: $y = e^{\alpha x}\left(c_1\cos\beta x + c_2\sin\beta x\right)$

$m = 2 \pm 3i$

Find the general solution of the given second-order differential equation.

Ex: $y'' - 10y' + 25y = 0$

Aux eqn: $m^2 - 10m + 25 = 0$

$(m-5)^2 = 0 \Rightarrow m = 5$, repeated

$$\boxed{y = c_1 e^{5x} + c_2 x e^{5x}}$$

Ex: $2y'' - 3y' + 4y = 0$

$2m^2 - 3m + 4 = 0 \qquad m = \frac{3 \pm \sqrt{9-32}}{4}$

$$m = \underbrace{\frac{3}{4}}_{\alpha} \pm \underbrace{\frac{\sqrt{23}}{4}}_{\beta} i$$

$$\boxed{y = e^{3/4 x}\left(c_1 \cos \frac{\sqrt{23}}{4}x + c_2 \sin \frac{\sqrt{23}}{4}x\right)}$$

Ex: Find the general solution of the given higher-order differential equation.

$\frac{d^3x}{dt^3} - \frac{d^2x}{dt^2} - 4x = 0$

$y = e^{mx}$

Aux eqn: $m^3 - m^2 - 4 = 0$

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$$m = 2, \ \underbrace{-\frac{1}{2}}_{\alpha} \pm \underbrace{\frac{\sqrt{7}}{2}}_{\beta} i$$

$$\boxed{x = c_1 e^{2t} + e^{-1/2 t}\left(c_2 \cos \frac{\sqrt{7}}{2}t + c_3 \sin \frac{\sqrt{7}}{2}t\right)}$$

Special 2nd-order differential equations:
$y'' + k^2 y = 0$ and $y'' - k^2 y = 0$, $k \in \mathbb{R}$

$m^2 + k^2 = 0$
$m = \pm ki$

$y = c_1 \cos kx + c_2 \sin kx$

$m^2 - k^2 = 0$
$m = \pm k$

$y = c_1 e^{kx} + c_2 e^{-kx}$

$c_1 = c_2 = \frac{1}{2} \Rightarrow y = \frac{e^{kx} + e^{-kx}}{2}$

$c_1 = \frac{1}{2},\ c_2 = -\frac{1}{2}$
$\Rightarrow y = \frac{e^{kx} - e^{-kx}}{2}$

$y = c_1 \cosh kx + c_2 \sinh kx$

Find a homogeneous linear differential equation with constant coefficients whose general solution is given.

Ex: $y = c_1 e^{-4x} + c_2 e^{-3x}$

$m_1 = -4,\ m_2 = -3$

$(m+4)(m+3) = 0 \Rightarrow m^2 + 7m + 12 = 0$

$$\boxed{y'' + 7y' + 12y = 0}$$

Ex: $y = c_1 e^{10x} + c_2 x e^{10x}$

repeated

Ex: $y = c_1 + c_2 e^{2x} \cos 5x + c_3 e^{2x} \sin 5x$

$m_1 = 0$ $\quad m_{2,3} = 2 \pm 5i$